

Research Statement

My research interests lie in algebraic topology and category theory, particularly in their intersection. My current and planned work is specifically in the subfield of parametrized (higher) category theory, and its applications to equivariant homotopy theory and Goodwillie calculus.

Since my first introduction to the fields, I've been allured by the symbiotic relationship between algebraic topology and category theory, where each one informs the growth of the other. The generalist nature of category theory provides the big picture ideas which help us figure out what ought to be the morally correct definitions. On the other hand, the process of understanding certain concrete phenomena in algebraic topology has mandated the development of new categorical foundations to capture them. These involve the invention of model categories, operads and ∞ -categories.

One recent subfield that has spawned to meet such a need is that of parametrized higher category theory. It was designed to encode certain insights that came from the solution to the Kervaire Invariant One problem by Hill, Hopkins and Ravenel in [HHR15]. One key idea was that the norms that appear in the category of equivariant spectra and the closely related notion of indexed (co)products aren't just extra structure that one needs to keep track of. Instead, they are integral to the structure of equivariant homotopy theory. As such, one would want a categorical framework where such data is baked in from the get-go.

While this was the motivating example, the theory of parametrized categories is fairly general and has had surprising applications to Goodwillie Calculus as well.

I will first talk about my current research and work in progress, and then talk more about a broader research program.

1 Current Research

Let us recall the classical story of Thom spectra as a motivating example. Let $p : E \rightarrow X$ be a rank n real vector bundle. One can fiberwise one-point compactify E to get an S^n bundle $S^E \rightarrow X$. The added points form a section of this bundle, which we can quotient out to construct the *Thom space* $\mathrm{Th}(E)$. Applying this to the universal rank n bundle $\gamma_n : \mathcal{E}_n \rightarrow BO(n)$, we get the spaces $\mathrm{Th}(\gamma_n)$ along with structure maps $\Sigma \mathrm{Th}(\gamma_n) \rightarrow \mathrm{Th}(\gamma_{n+1})$ corresponding to the functor $E \mapsto E \oplus \mathbb{R}$. This assembles to a *Thom spectrum*

$$\mathrm{MO} := (\mathrm{Th}(\gamma_n))_{n \in \mathbb{N}}$$

A more modern approach to Thom spectra was developed in [ABG⁺14]. From now on, we work in the context of ∞ -categories, which we just call categories. The reader should feel free to ignore the ∞ for the most part.

A spherical bundle $S^E \rightarrow X$ of the above sort gives a classifying map $X \rightarrow \mathrm{BGL}_1(S^n)$. We can compose this to get a map

$$X \rightarrow \mathrm{BGL}_1(S^n) \rightarrow \{n\} \times \mathrm{BGL}_1(\mathbb{S}) \rightarrow \mathbb{Z} \times \mathrm{BGL}_1(\mathbb{S}) \simeq \mathrm{Pic}(\mathbb{S}).$$

The Picard space of the sphere spectrum $\mathrm{Pic}(\mathbb{S})$ can also be thought of as the full subcategory of spectra spanned by the invertible spectra. We thus obtain a functor $f : X \rightarrow \mathrm{Sp}$. Then, the

associated Thom spectrum is the $\mathbf{Th}(f) := \operatorname{colim}(f)$. Since this was obtained from a vector bundle, we also have an equivalence $\mathbf{Th}(f) \simeq \Sigma^\infty \mathbf{Th}(E)$. The colimit should be thought as gluing copies of the sphere parametrized by X to obtain a “stable spherical bundle”. The Thom spectrum $\mathbf{MO}$ can similarly be obtained as the colimit associated to the J -homomorphism $J : \mathbf{BO} \rightarrow \mathbf{Pic}(\mathbb{S})$.

More generally, one can play the same game with any ring spectrum R . Let $\mathcal{S}$ be the category of spaces. The above construction assembles into a functor $\mathbf{Th} : \mathcal{S}_{/\mathbf{Pic}(R)} \rightarrow \mathbf{Mod}_R$. In practice, Thom spectra often carry certain multiplicative structure and one can ask how much of it is induced by this functor if it all. Barthel and Antolin-Camarena showed that the functor $\mathbf{Th}$ is sufficiently monoidal, so that if we start with a map $f : X \rightarrow \mathbf{Pic}(R)$ which is $\mathcal{O}$ -monoidal for some operad $\mathcal{O}$, then the Thom spectrum $\mathbf{Th}(f)$ is an $\mathcal{O}$ -algebra. Such information is useful as it provides universal properties for Thom spectra. For example, even though the ring spectra $H\mathbb{Z}/p$ are $\mathbb{E}_\infty$, they are only $\mathbb{E}_2$ Thom spectra and the corresponding universal property is essentially that $H\mathbb{Z}/p$ is the universal $\mathbb{E}_2$ ring spectrum with $p = 0$.

Categorically speaking, there should be a more moral reason why $\mathbf{Th}$ is monoidal. The category $\mathcal{S}_{/\mathbf{Pic}(R)}$ is equivalent to the category of presheaves on $\mathbf{Pic}(R)$ (which we denote by $\mathcal{P}(\mathbf{Pic}(R))$) by the straightening-unstraightening theorem of Lurie. The Thom spectrum functor can then be equivalently described as the left Kan extension

$$\begin{array}{ccccc} \mathcal{S}_{/\mathbf{Pic}(R)} & \simeq & \mathcal{P}(\mathbf{Pic}(R)) & \dashrightarrow & \mathbf{Mod}_R \\ & & \uparrow y & \nearrow & \\ & & \mathbf{Pic}(R) & & \end{array}$$

Since left Kan extensions are monoidal, the monoidality of $\mathbf{Th}$ boils down to proving the monoidality of straightening-unstraightening. This is something that “should” be true, and was worked out a few years later in much higher generality by Ramzi in [Ram26]. We state a relevant theorem of Ramzi’s:

Theorem (Ramzi). *Let $\mathcal{O}$ be an operad and $\mathcal{C}$ an $\mathcal{O}$ -monoidal category. The straightening/unstraightening equivalence can be enhanced to a $\mathcal{O}$ -monoidal equivalence*

$$\mathbf{Fun}(\mathcal{C}, \mathbf{Cat}_\infty) \simeq \mathbf{Cocart}_{\mathcal{C}}$$

where the domain has the Day convolution $\mathcal{O}$ -monoidal structure, and the target $\mathcal{O}$ -operad $(\mathbf{Cocart}_{\mathcal{C}})^\otimes$ is the full sub- $\mathcal{O}$ -operad of $((\mathbf{Cat}_\infty)_{/\mathcal{C}})^\otimes$ consisting of those multi-morphisms that preserve coCartesian edges in each variable.

The monoidality of $\mathbf{Th}$ then follows as a corollary.

Let us now move on to the equivariant analog. To that end, let G be a fixed finite group. Much like the classical setting, the equivariant analog of the classical Thom spectra play an important role in our understanding of the equivariant stable homotopy category. As described in the introduction, the correct setting for this is that of parametrized higher category theory, which was first developed in [BDG⁺16] and its sequels, though the initial ideas were present in [HH16]

We start off with an insight of Elmendorf, who observes that a G -space can equivalently be thought of as a functor $O_G^{op} \rightarrow \mathcal{S}$. We denote this category by $\mathcal{S}_G$. This guides our intuition throughout. The objects of study then are G -categories, which are functors $O_G^{op} \rightarrow \mathbf{Cat}$. The functors corresponding to $G/H \rightarrow G/K$ are called restrictions, and so the data of a G -category then is a collection of categories $\mathcal{C}_H$ with restriction functors. We refer to the whole gadget by $\underline{\mathcal{C}}$, and we use the underlines to consistently remind ourselves we are working with a “parametrized object”. The prototypical example is that of the G -category of G -spaces $\underline{\mathcal{S}}_G$ whose value at G/H is $\mathcal{S}_H$. We can similarly define the G -category of G -spectra $\underline{\mathcal{Sp}}_G$ and the G -category of G -categories $\underline{\mathcal{Cat}}_G$ and so on. It is in this context that G -spaces and G -spectra have the universal properties that one would hope for.

Our experience with equivariant algebra tells us that we should not just try to capture symmetric monoidal structures via a functor $O_G^{op} \rightarrow \mathbf{CAlg}(\mathbf{Cat})$. Instead we should also ask for transfers/norms. Thus the appropriate definition should be what one might call a symmetric monoidal Mackey functor. More precisely, a G -symmetric monoidal category is a product preserving functor from the category of spans of finite G -sets $\mathbf{Span}(\mathbb{F}_G) \rightarrow \mathbf{Cat}$. A G -operad is similarly a map over $\mathbf{Span}(\mathbb{F}_G)$ satisfying certain conditions.

With the basic set up worked out, we can mimic the constructions from earlier *mutatis mutandis*. Indeed, given an $\mathcal{Q}$ -ring spectrum $\underline{R}$, one can construct an equivariant Thom spectrum functor $\underline{\mathbf{Th}}: (\underline{\mathcal{S}}_G)_{/\mathbf{Pic}(\underline{R})} \rightarrow \mathbf{Mod}_{\underline{R}}$. The trouble, as earlier, is regarding the $\mathcal{Q}$ -monoidality of this functor. One would want to prove an equivariant analog of Ramzi’s theorem. In a simpler case – with $\mathcal{Q} = G\text{-}\mathbb{E}_\infty$ and using presheaves valued in $\underline{\mathcal{S}}_G$ instead of $\underline{\mathcal{Cat}}_G$ – was stated in [HHK⁺24] without proof and used to great effect aiding with certain computations of equivariant factorization homology.

While I was working on proving this, Zhu and Quinn in [QZ26] circumvented the issue somewhat by generalizing the work of [ACB19] to get multiplicative structures on equivariant Thom spectra. They did, however, need to use the conjectural statement for other purposes. Since then, in joint work with Zhu and Quinn, we have proven the more general conjectured result:

Theorem (G., Quinn, Zhu). *Let $\underline{\mathcal{Q}}^\otimes$ be a G - ∞ -operad and X be an $\underline{\mathcal{Q}}$ -monoidal G -space. Then, there is an equivalence*

$$(\underline{\mathcal{S}}_G)_{/X}^\otimes \simeq \underline{\mathcal{P}}(X)^{\otimes_{\text{Day}}}$$

of $\underline{\mathcal{Q}}$ -monoidal G -categories.

The above constitutes a substantial part of my thesis. An even stronger result (the actual analog of Ramzi’s theorem) is currently joint work in progress.

2 A program for the future

The wilderness of equivariant operads is not well understood. There are a few ways of importing classical operads into the equivariant world, and there’s an independent definition of $\mathbb{E}_V$ operads for a G -representation V . In practice, there is possibly only one other operad that shows up in classical homotopy theory: the Lie operad. Unlike the $\mathbb{E}_n$ operads, this is nonunital and makes sense only if the underlying category already has some additive structure, and is therefore best studied in the context of symmetric sequences in some sort of stable or additive category. Here’s an outline of the ways the in which the Lie operad and its variants connect to various (sub)fields:

1. **Goodwillie Calculus:** For any functor $F: \mathcal{S}_* \rightarrow \mathcal{S}_*$, its Goodwillie derivatives $\{\partial_n F\}$ form a symmetric sequence in spectra. When $F = id$, this is in fact the symmetric sequence associated to the (shifted) Lie operad.
2. **Genuine equivariant topology of Partition Complexes:** The n^{th} term in the symmetric sequence is equivalent to the dual of suspensions of the n^{th} partition complex. Arone and Brantner have used the genuine equivariance and combinatorics of the partition complexes to understand free Lie algebras in spectra in [AB21].
3. **Koszul Duality and Deformation Theory:** The Lie operad is Koszul dual to the $\mathbb{E}_\infty$ -operad. Over time, Lie algebras had also appeared in the context of deformation theory. Certain deformation problems were classified by elements satisfying the Maurer-Cartan equation in an associated DGLA. Lurie and Pridham unified these ideas to show that in characteristic 0, the Koszul duality functor induces an equivalence between formal moduli problems and spectral Lie algebras. Brantner and Matthew adapted these insights to make the story work in characteristic $p > 0$. They show that in positive characteristic, such moduli problems are equivalent to *Partition Lie Algebras* which one can morally think of as “divided power” Lie algebras [BM25].
4. **Intrinsic characterization:** Even if one doesn’t have all the fancy technology above, Heuts and Blans have an intrinsic characterization – the Lie operad is the unique (non-unital) operad such that the free algebra functor $\mathbf{Sp}^\otimes \rightarrow \mathbf{Alg}(\mathbf{Sp})^\wedge$ is (non-unital) symmetric monoidal [BH26].

It is worth pointing out that the cotangent complex/fibre is a gadget that appears in many of these connections and that these aren’t independent narratives, but part of cohesive whole. Given this, and my propensity to generalize things to the equivariant world, the following questions ask themselves.

Question 1. *Is there a good theory of Goodwillie calculus in the equivariant context?*

Dotto and Moi proposed one way one might go about it. While there are some incredible insights in their approach, there are some parts that feel lacking or not as genuine as one might want. More recently, Hilman and Linsens have a parametrized approach that looks promising. The setup of the theory is conceptually elegant, and builds on top of some ideas of Stoll. They have a good theory of excisiveness, but not of derivatives and cross-effects yet. This seems to something worth exploring.

Independently, Glasman proved that, classically, stable n -excisive functors can be classified by certain types of Mackey functors. This has already proven to be a fruitful way to think about Goodwillie calculus. Recently, Hilman, et al. refined Glasman’s theorem to a more parametrized setting. How can this be used to the benefit of ordinary Goodwillie Calculus?

Question 2. *What are the Koszul duals of the various N_∞ operads?*

Of course, this is conditional on Koszul duality working well for equivariant operads. There is a nice theory of (genuine) equivariant partition complexes by Bergner, et al. that should play a role here. One source of confidence in this theory is the close relationship to deformations. Hill shows that at the Tambara level, the cotangent space does a good job of controlling square zero

deformations. I’ve been informed that lifting it to equivariant TAQ is work in progress of some authors. This makes one hopeful about controlling higher order deformations in an analogous way.

Question 3. *What is the role of genuine Σ_n -equivariance in the classical Lie story and how can it be generalized?*

Strict quotients and fixed points are quite significant in the works of Arone, Brantner, Matthew, et al. Brantner et al. even use genuine Σ_n -spectra and genuine fixed points to model this behaviour. What is the extent of their utility? They should be generalizable to ∞ -categories with products and other categories built from such things, since strict commutative monoids do make sense there. Is this actually useful?

Question 4. *Can one formally deduce the existence of a G -operad that satisfies the universal property given by the “Intrinsic characterization” above?*

This is probably not that helpful. Nonetheless, the intrinsic characterization should be a guiding light for what we ought to construct.

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