Brown's Representability

Classical Thm: $F : \mathrm{Ho}\,\mathrm{CW}_*^{op} \longrightarrow \mathrm{Set}$ preserves products, weak pullbacks. Then F is representable

$H^n(X;A) \cong [K(A,n), -]$ i.e., $FX \cong [A, X]$

Setting: $\mathcal{C}$ is a P-cat if it has products and weak pullbacks

$F : \mathcal{C} \to \mathrm{Set}$ is called half-exact products and weak pullbacks.

i) $\mathcal{C}$ has weak limits

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d] & X \arrow[d, "g"] \\ X \times Y \arrow[r, "f"'] & Y \end{tikzcd}$$

ii) F preserves some weak pullback then it'll preserve any weak pullback

Defn.: $\mathcal{A} \subset \mathcal{C}$ is right cardinally bounded if $\exists$ large enough $\mathcal{B}$

^{op} $\mathcal{C}$ $\bar{X}$

$X : \beta^{op} \to \mathcal{C}$, X

$$\operatorname*{colim}_{\alpha<\beta} \mathcal{C}(X_\alpha, Y) \xrightarrow{\simeq} \mathcal{C}(\bar{X}, Y)$$

for $Y \in \mathcal{A}$.

$$\begin{tikzcd}
 & & & \bar{X} \arrow[dlll] \arrow[dll] \arrow[dl] \arrow[dr] & \\
X_0 & X_1 \arrow[l] & X_2 \arrow[l] & \cdots & X_\alpha \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

<u>Defn</u>: $\mathcal{A} \subset \mathcal{C}$ is said to right adequate
$Y \in \mathcal{A}$ $\{\mathcal{C}(-, Y)\}$ reflects iso.

$\mathcal{C}(f, Y)$ iso $\Rightarrow$ f is an iso.

<u>Defn</u>: A solution set for F is
a set S s.t. for any $X \in \mathcal{X}$
and $x \in FX$, $\exists\, Y \in S$ and $y \in FY$
s.t. $\exists\, f : Y \to X$
$y \mapsto x$

$\int F$ $\quad (X, x)$, $x \in FX$ $\quad (X, x) \to (Y, y)$
$X \to Y$
$x \mapsto y$.

<u>Thm</u>: $F : \mathcal{C} \to \mathrm{Set}$, $\mathcal{C}$ has a RCB

and right set $\mathcal{A}$, then
F is representable.

<u>Def^n:</u> F is hyporep. if F is a retract
of $\mathcal{C}(X, -)$

$$FY \xrightarrow{\alpha_Y} \mathcal{C}(X, Y) \xrightarrow{\beta_Y} FY$$

<u>Thm:</u> F is hyporepresentable iff
it is half-exact and SS

<u>Prop:</u> i) Representable iff $\int F$
ii) Hyporentable iff $\exists X' \in \int F$
s.t. $\int F(X', -)$ $\quad$ $\int F(X', -)$

$$\underline{\int F(X', Y') \ni f} \qquad \uparrow$$

Pf ii)

$$\underset{y}{FY} \xrightarrow{\alpha_Y} \underset{f}{\mathcal{C}(X, Y)} \xrightarrow{\beta_Y} FY$$

$X' = (X, ?)$
$Y' = (Y, y)$

$$\begin{tikzcd}
 & X \arrow[r] & Y \\
 & f \arrow[r, mapsto] & y \\
FY^{y} \arrow[r, "\alpha_Y"] & \mathcal{C}(X, Y) \arrow[r, "\beta_Y"] & FY & \longleftrightarrow \\
FX \arrow[u, "Ff"] \arrow[r, "\alpha_X"] & \mathcal{C}(X, X) \arrow[u, "f_*"] \arrow[r, "\beta_X"] & FX \arrow[u, "Ff"] \\
x & id \arrow[r, mapsto] & x
\end{tikzcd}$$

$$F f x = y$$

$$f_y : (X, x) \longrightarrow (Y, y)$$

$\mathcal{C}$ P-category $\qquad \underline{\Pi(X, x) = X}$

F half-exact.

$$\Pi(A_i, a_i)$$

$$\int F \ni \{(A_i, a_i)\}_{i \in I} \qquad \left(\prod_{i \in I} A_i, a\right)$$

$$\begin{tikzcd}
F \prod A_i \arrow[r] & A_i \\
\{\cdot\} \arrow[u, "a"] \arrow[ur, "a_i"'] &
\end{tikzcd}$$

$\underline{\lambda : D \to \int F}$ $\qquad \lambda d = (Y, \underline{y})$

$$\begin{tikzcd}
& & \lambda d_2 \arrow[d] \\
(X, x) \arrow[r] \arrow[urr] \arrow[drr] & \lambda d_1 \arrow[ur, dashed] & \lambda\lambda_1
\end{tikzcd} \quad \cdots$$

$$\Pi \lambda : D \to \mathcal{C}$$

$$\mathrm{colim}\ \Pi \lambda \cong X$$

$F\Pi\lambda : D \to \mathrm{Set}$, $\quad \pi_2(X,x) \ni x$

$$\begin{tikzcd}
\operatorname{wlim} F\Pi\lambda \cong FX \\
\{*\} \arrow[u, "\pi_2\lambda"]
\end{tikzcd}
\qquad
\begin{tikzcd}
FX \arrow[r, "\mu_d"] & F\Pi\lambda d \\
\{*\} \arrow[u, "x"] \arrow[ur, "\pi_2\lambda d"'] &
\end{tikzcd}$$

$(X,x) \rightsquigarrow \lambda$

$$\begin{tikzcd}
X \arrow[r, "h"] & A \arrow[r, shift left=3, "f"] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] \arrow[r, bend right=40, "g"'] & B
\end{tikzcd} \quad \text{in } \int F$$

$fh = gh$

Th$^{\text{m}}$: F hyporep iff $\underline{SS}$ and half-exact

$(\Rightarrow)$ $FY \xrightarrow{\alpha_Y} C(\underline{X},Y) \to FY$

$\{x\}$

$Y = \Pi Y_i$

$$\begin{tikzcd}
F\Pi Y_i \arrow[r, "\alpha"] \arrow[d, "h"'] \arrow[drr, bend left=10, "k"] & C(X,\Pi Y_i) \arrow[r, "\beta"] \arrow[d, "?"] & F\Pi Y_i \arrow[d, "h"] \\
\Pi F Y_i \arrow[r, "\alpha"] \arrow[rr, bend right=30] & \Pi C(X,Y_i) \arrow[r, "\beta"] \arrow[ur, bend left=10, "g"] & \Pi F Y_i
\end{tikzcd}$$

$hg = id_{\Pi F Y_i}$ ✓

$gh = id_{F \Pi Y_i}$

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

$$\begin{tikzcd} & 2 \arrow[d] \arrow[r, bend left] & FY \arrow[r] \arrow[d] & F\cdot \arrow[d] \arrow[ddl, bend left, "\alpha."] \\ & & F\cdot \arrow[r] \arrow[ddll, bend right, "\alpha."] & F\cdot \\ & C(X,Y) \arrow[r] \arrow[d] \arrow[uur, "\beta_Y"] & C(X,\cdot) \arrow[d] & \\ & C(X,\cdot) \arrow[r] & C(X,\cdot) & \end{tikzcd}$$

$(\Leftarrow)$ $\mathcal{A}$ ss.

$A \in \mathcal{A} \subset \mathcal{C}$

$\mathcal{B} = \{(A,a)\} \subset \int F$

$B = \Pi (A,a) \xrightarrow{p} (X,x)$

$f : A \longrightarrow *$

$a \mapsto x$

$C \xrightarrow{c} B \rightrightarrows B$

$(F(C,W)) \to^{B}$ $\qquad c = 2bc$

$$\begin{tikzcd} 2 \arrow[r, shift left, "2"] & B \arrow[l, shift left, "b"] \arrow[r, shift left, "w"] \arrow[r, shift right, "\hat{w}"'] & W \\ & C \arrow[u, "c"] \arrow[ur] & \end{tikzcd}$$

$wc = \hat{w}c$

$$wc = w2bc = \hat{w}2bc = \hat{w}c$$

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \int F(C,W) \arrow[rr] & & \int F(C,W') \end{tikzcd}$$

wc $\qquad$ $w'c$

$\hat{w}'c$

W, W'

$W \to W'$

Brown's Rep.

Thm: $\mathcal{C}$, $F : \mathcal{C} \to \mathrm{Set}$, Then F is rep.

$\mathcal{C}$ RCB and Right adequate set

$$\left[X : \beta^{op} \to \mathcal{C} \quad \middle| \text{ Reminder} \right.$$

$$\left. \operatorname*{colim}_{\alpha<\beta} \mathcal{C}(X_\alpha, Y) \cong \mathcal{C}(\bar{X}, Y) \right]$$

Proof: $\mathcal{A} \subset \mathcal{C}$, $\mathcal{B} \subset \int F = \{(A, a)\}$

$B = \Pi(A, a)$

$X \in \int F$

$\bar{X}$

$$\frac{Y : \beta^{op} \longrightarrow \int F}{Y_0 = X \times B}$$

$Y_{\alpha+1} \longrightarrow Y_\alpha \rightrightarrows Y_0$

$\gamma < \beta$

$\tilde{Y} : \gamma^{op} \longrightarrow \int F$

Y_γ a cone over $\tilde{Y}$

$\Pi \bar{X}$.

$\bar{X} \longrightarrow Y$

$\exists!\ \bar{X} \longrightarrow (A, a)$

$\bar{X} \xrightarrow{f} (A,a)$

$\bar{X} \longrightarrow Y_0 = X \times \Pi(A, a)$

ϕ

Y_r

$\hat{f}$

$$\operatorname{colim}_{\alpha<\beta} \mathcal{C}\left(\prod Y_\alpha, A\right) \cong \mathcal{C}\left(\prod \bar{X}, A\right)$$

$$f = \tilde{f} k g = \sim k g = \emptyset$$

$$\exists!\ \bar{X} \to (A, a)$$

$$\bar{X} \cong \bar{Y}$$

$$\mathcal{C}(\bar{X}, A) \Rightarrow \mathcal{C}(\bar{Y}, A)$$

$$\bar{X} \to X$$

$$\begin{tikzcd} \overline{\bar{X} \times Y} \arrow[r] & \bar{X} \times Y \arrow[d] \\ & \bar{X} \arrow[ul, bend left] \end{tikzcd}$$

$$\bar{X} \to \bar{X} \times Y \to Y \in \int F$$

$\bar{X} \to Y$ $\qquad$ $\bar{X}$ is initial in SF

$c(\bar{X}, -)$

$$\begin{tikzcd}
Z \arrow[r, "z"] & \bar{X} \arrow[r, shift left] \arrow[r, shift right] & Y \\
\bar{Z} \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

$\exists !\, ;\ \bar{X} \to Y$