Model Categories

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Preface

The following set of notes is a product of a reading project under Prof. Amit Kuber at IIT Kanpur, whose guidance has been invaluable in the making of these notes, as he helped me with understanding the material and provided feedback on the writing and contents of these notes.

The main goal is to provide a quick run-through of Model Categories and some discussion of the Model Structure on **Top** and its connection with the Model Structure on the category of simplicial sets. The first chapter, providing motivational prerequisites from topology, is inspired by Peter May's *A concise course in Algebraic Topology*[2] and the rest of the content for these notes is heavily inspired by Mark Hovey's *Model Categories*[1].

Additional references for the topics are provided in the bibliography, [3] is another popular book on Model Categories, [5] includes classical homotopy theory and [4] is for further reading.

Chapter 1

Motivation: Some classical homotopy theory

1.1 A convenient category for homotopy theory

One of the main issues we encounter when we work with **Top** is that the following does not hold:

$$\mathbf{Top}(X \times Y, Z) \simeq \mathbf{Top}(X, Z^Y)$$

To rectify this, we work with a subcategory that we shall call **CGTop**. First, some preliminary definitions:

Definition 1.1. We call a space X a k-space if for every compact Hausdorff space K, and every map $g : K \to X$, $g^{-1}(A)$ closed in $K \implies A$ is closed in X.

Definition 1.2. We call a space X *weak Hausdorff* if for every compact Hausdorff space K, and every map $g : K \to X$, $g(K)$ is closed in X.

The objects in **CGTop** are spaces which are both k-spaces and *weak Hausdorff*. Morphisms are just continuous maps between them. Note that Haudorff spaces, locally compact space are inside **CGTop**.

We have the *k-ification* of a topological space X, called kX, defined as follows: kX is the same as X as a set. A subset of kX is closed if its intersection with compact Hausdorff subsets of (the original topology on) X is closed (in the original topology on X).

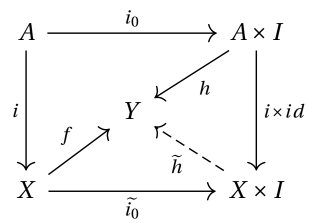
We denote the product of X and Y in this new category by $X \times_{CG} Y = k(X \times Y)$. We also imbue the function space the *k-ification* of the compact-open topology. Finally, we have the following property:

$$\mathbf{CGTop}(X \times_{CG} Y, Z) \simeq \mathbf{CGTop}(X, Z^Y)$$

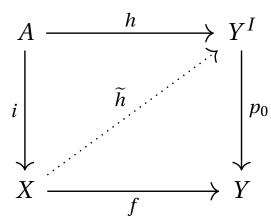
We shall now denote $\times_{CG}$ simply by $\times$ as we will exclusively work with **CGTop**. In particular, a homotopy $X \times I \to Y$ is simply a map $X \to Y^I$.

1.2 Cofibrations

Definition 1.3. A map $i : A \to X$ is called a cofibration if for every space Y, and maps f, h making the following diagram (a) commute, there exists a map $\widetilde{h}$ such that the diagram retains commutativity.



(a)



(b)

(b) portrays an equivalent formulation using the new way of writing a homotopy, as described above.
Now, consider the mapping cylinder

$$Mi = X \cup_i A \times I$$

Replacing Y in the above diagram with Mi, and f and h with the respective pushout maps, we see that if there exists a map $r : X \times I \to Mi$ making the diagram commute, then i is a cofibration since for any Y and f and h as above, by the pushout property, there exists a map $Mi \to Y$ whose composite with r gives us the required map. Another important fact, whose proof we omit, is that cofibrations are *exactly* closed inclusions where the subspace is a **Neighbourhood Deformation retract(NDR)**. That is, there exist a neighbourhood $A \subset V \subset X$, such that V deformation retracts to A.
This leads us to the following lemma:

Lemma 1.1. *Every map can be expressed as a composition of a cofibration and a homotopy equivalence.*

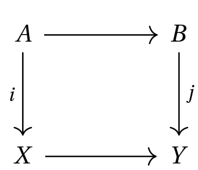
Proof. For a map $f : X \to Y$, it can be checked that the following splitting satisfies the above criterion

$$X \xrightarrow{j} Mf \xrightarrow{r} Y$$

where $j(x) = (x, 1)$ and $r(x, s) = f(x)$ and $r(y) = y$ □

Another lemma, which we shall see in a different setting, but omit the proof here:

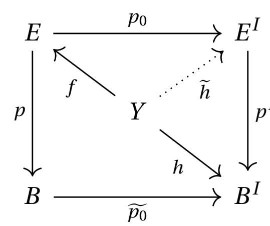
Lemma 1.2. *Cofibrations are closed under pushouts. That is, in the pushout diagram, if i is a cofibration, so is j.*



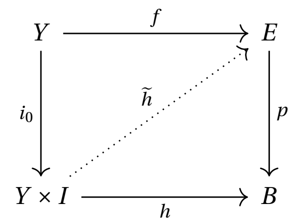
1.3 Fibrations

Most of the statements in this section are analogues of ones above, so we leave out quite a few details.

Definition 1.4. A surjective map $p : E \to B$ is called a fibration if for every space Y, and maps f, h making the following diagram (a) commute, there exists a map $\widetilde{h}$ such that the diagram retains commutativity.



(a)



(b)

A construction dual to that of the mapping cylinder is the mapping path space

$$Np = E \times_p B^I$$

and now we care about the existence of a function $Np \to E^I$.

Example 1.1. Fiber bundles, in particular, covering spaces, are fibrations.

Lemma 1.3. *Every map can be expressed as a composition of a homotopy equivalence and a fibration.*

Proof. For a map $f : X \to Y$, the splitting in this case is

$$X \to Nf \to Y$$

□

Lemma 1.4. *Fibrations are closed under pullbacks.*

We now have a nice theorem relating the two notions that we have just explored.

Theorem 1.1. *If $i : A \to X$ is a cofibration, B is a topological space, then the map*

$$B^i : B^X \to B^A$$

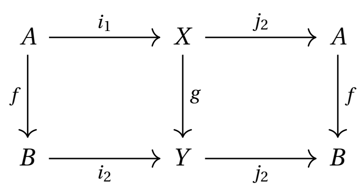
is a fibration.

Chapter 2

Model Categories

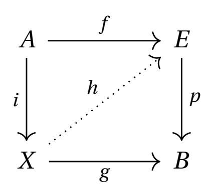
Some preliminary definitions:

Definition 2.1. Let $f : A \to B$, $g : X \to Y$ be morphisms in a category **C**. f is called a **retract** of g if there exist maps i_1, i_2, j_1, j_2 such that the following diagram commutes and the horizontal compositions are the identity morphisms.



In particular, in the category Map **C**, where objects are morphisms of **C**, and morphisms of Map **C** are of the form $i = (i_1, i_2)$ as above, we have $i : f \to g$ and $j : g \to f$ such that $j \circ i = id_f$.

Definition 2.2. We say the morphisms i has the **left lifting property(LLP)** with respect to a morphism p or equivalently that p has the **right lifting property(RLP)** with respect to i if for all morphisms f, g such that the diagram commutes, there exists a lift h such that the diagram retains commutativity.



Definition 2.3. (α, β) is called a functorial factorization for **C** if α, β are functors Map **C** $\to$ Map **C** such that for every morphism f,

$$f = \beta(f) \circ \alpha(f)$$

With this, we can finally begin to define a model category.

Definition 2.4. A **model structure** on a category **C** is a collection of three subcategories **W**, **F** and **COF**, (which stand for weak equivalences, fibrations and cofibrations respectively) along with two functorial factorizations (α, β) and (γ, δ) which satisfy the following properties:

- (2-out-of-3 property) If for morphisms f and g, fg is defined and if two out of f, g and fg are weak equivalences, then so is the third.
- **W**, **F**, **COF** are closed under retracts
- **Trivial cofibrations** (morphisms which are both weak equivalences and cofibrations) have the LLP with respect to all fibrations. Similarly, cofibrations have the RLP with respect to all **trivial fibrations**.
- For every morphism f, $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration and $\delta(f)$ is a fibration.

C is said to be a **model category** if it has a model structure and it admits small limits and colimits.

A simple consequence is that initial and terminal objects exist in a model category.

Example 2.1. For any category **C** with small limits and colimits, we can let one of the classes of morphisms be all isomorphisms and the rest are all morphisms.

Example 2.2. For **Set**, we can consider one possible model structure: Every map is a weak equivalence, cofibrations are exactly all injections and fibrations are exactly all surjections.

For any model category **C**, we have the dual category D**C**, with the following dual structure: weak equivalences are the same in both categories, cofibrations and fibrations are swapped. α and δ are swapped, so are β and γ. So, theorems proven for cofibrations can be dualized for fibrations.

Definition 2.5. An object X is called **cofibrant** if the unique map from the initial object is a cofibration. Dually, an object Y is called a **fibration** if the unique map to the terminal object is a fibration.

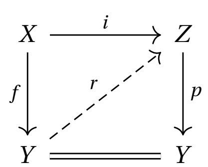
Given a map $0 \to X$, we can split it into a cofibration followed by a trivial fibration $0 \to QX \to X$. The functor Q is called the cofibrant replacement functor. Similarly, we can define R, the fibrant replacement functor.

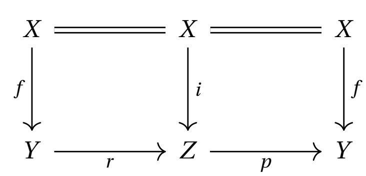
One encounters Top_* quite frequently in homotopy theory, so it is somewhat natural to ask for a model structure induced on $\mathbf{C}_*$ [1] by a model structure on $\mathbf{C}$. It turns out to be pretty intuitive: f is a weak equivalence/ fibration/ cofibration iff Uf is a weak equivalence/ fibration/ cofibration respectively, where $U : \mathbf{C}_* \to \mathbf{C}$ is the forgetful functor sending (X, v) to X.

The following lemma is simple to prove, but turns out to be quite useful.

Lemma 2.1 (Retract Argument)**.** *If $f = pi$ and f has the **LLP** with respect to p, then f is a retract of i. Dually, if f has the **RLP** with respect to i, then f is a retract of p.*

Proof. The first part is immediate from the following diagrams:





□

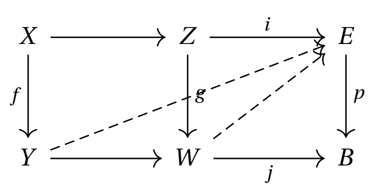
One interesting thing to note is that the definition of a model structure is overdetermined. Indeed, we can define the class of cofibrations and trivial cofibartions using fibrations and trivial fibrations. This is encompassed in the following lemma:

Lemma 2.2. *f is a cofibration iff f has the **LLP** with respect to all trivial fibration. Dually, f is a trivial cofibration iff f has the left lifting property with respect to all fibrations.*

Now, we prove something that we promised in the first chapter.

Lemma 2.3. *Cofibrations are closed under pushouts*

Proof. Say f is a cofibration and g is a pushout of f. If p is a trivial fibration, and we have i and j such that the following diagram commutes,



then we have a lift $Y \to E$ since f is a cofibration. The lift $W \to E$ exists by the pushout property. Hence g has the **LLP** with respect to p. □

[1] where $\mathbf{C}_*$ the category consisting of objects of the form (X, v) where X is an object of $\mathbf{C}$ and v (which is called a basepoint) is a morphism from the terminal object to X. One can see how this works out in the case where $\mathbf{C} = \mathbf{Top}$.

We now have an important lemma due to Ken Brown. This works in a slightly more general setting, however we will settle for the case where our category is a model category.

Lemma 2.4 (Ken Brown)**.** *Suppose* **C** *is a model category and* **D** *is a category with a subcategory of weak equivalences which satisfies the two out of three axiom. Suppose $F : C \to D$ is a functor which takes trivial cofibrations between cofibrant objects to weak equivalences. Then F takes all weak equivalences between cofibrant objects to weak equivalences. Dually, if F takes trivial fibrations between fibrant objects to weak equivalences, then F takes all weak equivalences between fibrant objects to weak equivalences.*

Chapter 3

The Homotopy Category

An isomorphim is the standard notion of an equivalence. What should one do if one wants to introduce a new notion of equivalence, and study the objects under consideration under this new notion? A standard example, of course, is studying topological spaces upto (weak) homotopy equivalences.
Our strategy will be to construct a new category where these "weak" equivalences are in fact isomorphisms.
So given a model category **C**, we construct the homotopy category **HoC** by considering the free category generated by the morphisms in **C** and w^{-1} for every w in **W**. Essentially, **HoC** has the same objects as **C** and its morphisms are chains alternating between morphisms in **C** and w^{-1}. Let $\gamma : \mathbf{C} \to \mathbf{HoC}$ be the functor which is identity on the objects and sends morphism to the corresponding singleton chains.

HoC is in some sense the "smallest" category where we have our weak equivalences inverted. In fact, the isomoprhisms in **HoC** are exactly weak equivalences in **C** or their inverses. This is captured in the following lemma, which captures the universal property:

Lemma 3.1.

- *If $F : C \to D$ is a functor which sends weak equivalences to isomorphisms, then $\exists!\ HoF : \mathbf{HoC} \to \mathbf{HoD}$ such that $HoF \circ \gamma = F$.*
- *If $\delta : \mathbf{C} \to \mathbf{E}$ is a functor which sends weak equivalences to isomorphisms and satisfies the above universal property, then $\exists!\ F : \mathbf{HoC} \to \mathbf{E}$ such that $F\gamma = \delta$.*
- *There exists a categorical isomorphism $\mathbf{D}^{\mathbf{HoC}} \cong$ the subcategory of $\mathbf{D}^{\mathbf{C}}$ with functors which send weak equivalences to isomorphisms.*

We denote by $\mathbf{C_c}$ the subcategory of cofibrant objects. We similarly define $\mathbf{C_f}$ and $\mathbf{C_{cf}}$. The next lemma tells us that the study of **HoC** can often be reduced to the study of a simpler subcategory.

Lemma 3.2. *The following inclusions are categorical equivalences*

$$\mathbf{HoC_{cf}} \longrightarrow \mathbf{HoC_c} \longrightarrow \mathbf{HoC}$$

$$\mathbf{HoC_{cf}} \longrightarrow \mathbf{HoC_f} \longrightarrow \mathbf{HoC}$$

A different approach would be to define a notion of homotopy in our model category, which will help us define a homotopy equivalence. Results and definitions here will closely mirror the ones that we see in the case of topological spaces.

Definition 3.1.

- A **cylinder object** for B is a factorization of the codiagonal map $B \sqcup B \to B$ into $B \sqcup B \xrightarrow{i_0+i_1} B' \xrightarrow{s} B$, where the first map is a cofibration and the second is a weak equivalence. The canonical cylinder object which we get from one of the functorial factorizations is denoted by $B \times I$; this is what we are used to when it comes to topological spaces.
- A **path object** for X is a factorization of the diagonal map $X \to X \times X$ into $X \xrightarrow{r} X' \xrightarrow{p_0,p_1} X \times X$, where the first map is a weak equivalence and the second is fibration. Analogously, The canonical path object is denoted by X^I

With this, we can go on to define left and right homotopies. Let f and g be two maps $B \to X$.

Definition 3.2.

- A **left homotopy** between f and g is a morphism $H : B' \to X$ such that $Hi_0 = f$ and $Hi_1 = g$.
- A **right homotopy** between f and g is a morphism $K : B \to X'$ such that $p_0K = f$ and $p_1K = g$.

Obviously, these are dual notions, so it is enough for us to focus on one of these.
Left homotopies satisfy properties that we expect them to (at least in most cases). They are preserved under left composition, and when X is fibrant, they are preserved under right composition. When B is cofibrant, this is an equivalence relation and gives a right homotopy between f and g.

These results give us the right to talk about the quotient category $\mathbf{C_{cf}}/\sim$ under the homotopy equivalence relation. As one would want, we have the fact that the notions of homotopy equivalences and weak equivalences coincide in $\mathbf{C_{cf}}$, albeit not in $\mathbf{C}$.

We summarize some of the above results and add some more in the next theorem:

Theorem 3.1.

- *The inclusion* $\mathbf{C_{cf}} \to \mathbf{C}$ *induces the categorical equivalence*

$$\mathbf{C_{cf}}/\sim \cong \mathbf{HoC_{cf}} \cong \mathbf{HoC}$$

- *There are natural isomorphisms*

$$\mathbf{C}(QRX, QRY)/\sim \cong \mathbf{HoC}(\gamma X, \gamma Y) \cong \mathbf{C}(RQX, RQY)/\sim$$

- $\gamma : \mathbf{C} \to \mathbf{HoC}$ *identifies left homotopic maps*
- *If* f *is morphism in* $\mathbf{C}$ *such that* γf *is an isomorphism, then* f *is a weak equivalence.*

Chapter 4

Quillen Functors and Quillen adjuctions

Quillen functors and Quillen adjuctions are functors and adjunctions which respect the homotopic structures of model categories. These will often help us study homotopic structures of a certain category in a different setting, which could make the problem simpler or offer more insight.

Definition 4.1. Let **C** and **D** be model categories.

- A functor $F : \mathbf{C} \to \mathbf{D}$ is called a **left Quillen functor** if F is a left adjoint and preserves cofibrations and trivial cofibrations.
- A functor $U : \mathbf{D} \to \mathbf{C}$ is called a **right Quillen functor** if U is a right adjoint and preserves fibrations and trivial fibrations.
- An adjunction $(F, U, \phi) : \mathbf{C} \to \mathbf{D}$ is called a **Quillen adjunction** if F is a left Quillen Functor.

The adjunction induces the unit map $\eta_X : X \to UFX$ and the counit map $\epsilon_Y : FUY \to Y$, which we can use to prove that if (F, U, ϕ) is a Quillen adjunction, then U is a right Quillen functor.
We have can compose Quillen adjunctions, namely if we have the Quillen adjunctions (F, U, ϕ) and (F', U', ϕ') their composition is

$$(F' \circ F, U \circ U', \phi \circ \phi')$$

With composition defined, we can see that in some sense, these act as morphisms of model categories.

We now want to look at how Quillen functors induce functors between the homotopy categories.

Definition 4.2. If $F : \mathbf{C} \to \mathbf{D}$ is a left Quillen functor, we define the **total left derived functor** $LF : \mathbf{HoC} \xrightarrow{HoQ} \mathbf{HoC_c} \xrightarrow{HoF} \mathbf{HoD}$.
Given a natural transformation $\tau : F \to F'$, we define the **total left derived transformation** $L_\tau = Ho\tau \circ HoQ$. That is, $(L_\tau)_X = \tau_{QX}$.

We can similarly define the dual notions of total right derived functors and transformations, which we denote by R.
While the total left left derived transformation is functorial, the total left derived functor is not. However it is "almost functorial," in the sense that it is associative upto a natural isomorphism. They also induce an adjunction between homotopy

categories given an adjunction between two model categories, that is, $L(F,U,\phi) = (LF, RU, R\phi)$.

Definition 4.3. A Quillen adjunction (F,U,ϕ) is called a **Quillen equivalence** if for all cofibrant X in **C** and fibrant Y in **D**, $f : FX \to Y$ is a weak equivalence iff $\phi(f) : X \to UY$ is a weak equivalence.

If we want the notion of a Quillen equivalence to be an "equivalence upto homotopy", then we should expect the induced adjunction between the homotopy categories to be a categorical equivalence. Indeed, this is captured in the following lemma, which also gives a another characterization of a Quillen equivalence.

Lemma 4.1. *The following are equivalent*

- (F,U,ϕ) *is a Quillen equivalence.*
- $X \xrightarrow{\eta_X} UFX \xrightarrow{Ur_{FX}} URFX$ *is a weak equivalence for all cofibrant* X *and* $FQUY \xrightarrow{Fq_{UY}} FQY \xrightarrow{\epsilon_Y} Y$ *is a weak equivalence for all fibrant* Y.
- $L(F,U,\phi)$ *is an adjoint equivalence of categories.*

The above lemma implies that we can refer to a Quillen equivalence just by its associated left Quillen functor (or the right Quillen functor). One can check that Quillen equivalences satisfy the 2-out-of-3 property, lending more credence to the notion of Quillen adjunctions as morphisms of model categories and Quillen equivalences as weak equivalences.

We end this chapter with a criterion for L_τ to be a natural isomorphism.

Lemma 4.2. *Suppose* $\tau : F \to G$ *is a natural transformation between left Quillen functors. Then,* L_τ *is a natural isomorphism iff* τ_X *is a weak equivalence for all cofibrant* X.

Chapter 5

Simplicial Sets

The goal of this chapter is to give a glimpse of the model structure of simplicial sets, which turns out to be Quillen equivalent to **Top** where the weak equivalences are exactly the weak homotopy equivalences. This is useful since simplicial sets are combinatorial in nature and are often simpler to work with than topological spaces.

Definition 5.1. We have the category Δ where objects are sets of the form $\{0, 1, \ldots, n\}$ which we denote by $[n]$. The morphisms are weakly order preserving maps. We have the **coface** maps which are the injective morphisms $d^i : [n-1] \to [n]$ which don't have i in their image. Similarly, we have the **codegeneracy** maps which are the surjective morphisms $s^i : [n] \to [n-1]$ which send i and $i+1$ to the same element.

One can show that any morphism can be written as as a composition of coface maps followed by a composition of codegeneracy maps.

For any category **C**, we have the category of simplicial objects $\mathbf{C}^{\Delta^{\mathbf{op}}}$. In particular, if $\mathbf{C} = \mathbf{Set}$, we call the it the category of simplicial sets, denoted by **SSet**.

For a simplicial set K, we denote $K[n]$ by K_n, which we call the n-simplices of K. For any element of K_n, we say that its dimension is n. We have the face maps $d_i : [n] \to [n-1]$ and the degeneracy maps $s_i : [n-1] \to [n]$. These are analogous to the maps we define for simplicial complexes in **Top**.

A simplex obtained from the successive applications of face maps to x is called a face of x. Similarly, a simplex obtained by the successive applications of degeneracy maps to x is called a degeneracy of x. For any $x \in K_n$, there exists a unique y of least dimension such that y is a degeneracy of x.

Example 5.1. For each $n \in \mathbb{N}$, we have the simplicial set $\Delta[n]$ which sends $[k]$ to $\Delta([k], [n])$.

Keeping simplicial complexes in mid we can define the boundary $\delta\Delta[n]$ whose nondegenerate k-simplices correspond to nonidentity injective order-preserving maps $[k] \to [n]$. One can similarly define the r-horn $\Lambda^r[n]$

We now have a lemma which will finally help us construct an adjunction that turns out to be the Quillen equivalence that we mentioned above.

Lemma 5.1. *Suppose* **C** *has small colimits. Then there exists a categorical equivalence* $\mathbf{C}^{\Delta} \cong$ *category of adjunctions* **SSet** $\rightarrow$ **C**.

We have a functor $\Delta \rightarrow$ **Top** which sends $[n]$ to the standard n-simplex. The above lemma tells us that there is a corresponding adjunction $(|\ |, Sing, \phi)$. We call $|\ |$ the geometric realization functor.

We briefly define the model structure on **SSet**, though we will be unable to prove that it is in fact a model structure.

Definition 5.2.
$I = \{\delta\Delta[n] \rightarrow \Delta[n] : n \geq 0\}$, where the maps are the canonical injections.
$J = \{\Lambda^r[n] \rightarrow \Delta[n] : n > 0, 0 \leq r \leq n\}$, where the maps are the canonical injections.

The class of fibrations is the class $J - inj$, that is, the class of maps that has the **RLP** with respect to every map in J.
The class of cofibrations is the class $(I - inj) - proj$, that is, the class of maps that has the **LLP** with respect to every map in $I - inj$.

While we are unable to prove that the adjunction above is indeed a Quillen equivalence, we provide the following observations to support our statement:
f is a weak equivalence in **SSet** iff the geometric realization $|f|$ is a weak equivalence in **Top**.

We have that $|\Delta[n]| = \mathbb{D}^n$ and that $|\delta\Delta[n]| = S^{n-1}$, so we can see that the maps in $J - inj$ correspond to Serre fibrations (maps which have **RLP** with respect to $S^{n-1} \hookrightarrow \mathbb{D}^n$) which are the fibrations in the model structure for **Top** that we are considering. Similarly maps in $(I - inj) - proj$ correspond to the cofibrations in the model structure for **Top**.

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