K-Theory, Bott Periodicity and the Index Theorem

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0 Introduction

While we have seen an analytic approach to the Index Theorem, there is another approach to it using methods from algebra and topology. This approach, which uses K-theory, can often be more insightful and conceptual, though the heat kernel approach brings it's own unique geometric perspective. It is a curious mathematical phenomenon as to how two wildly different methods can be powerful enough to prove the the powerful Index Theorem!

The Bott Periodicity theorem comes in many forms and is of varying interests to mathematicians. While Bott's original formulation of the theorem was about the periodicity of homotopy groups of the infinite unitary, orthogonal and spin groups, the modern formulation of the complex (unitary) case helps us prove that (complex) K-Theory can be made into a generalized cohomology theory. The periodicity theorem rears its head multiple times in the study of index theory, and we'll try to give a good picture of how it comes up.

We'll look at a K-theoretic formulation of the Index Theorem, and see how it relates to the classical statement. The K-theoretic restatement is much more pleasing to the eye, and the translation to the classical version will also show us how the Todd genus comes into play.

Even though we start from the definitions of K-Theory, we will be liberal with the use of some facts outside of the scope of this essay, as long as it aids in motivating the concepts at hand. We'll also assume some knowledge of basic homotopy theory. This essay is not supposed to be rigorous or detailed, but aims to provide a broad outline of the techniques and ideas used in this approach.

1 Complex K-Theory

Recall that for a compact Hausdorff topological space X, the 0^{th} K-group or its K-theory is

$$K(X) = K^0(X) = G(Vect(X)).$$

Here, $Vect(X)$ is the set of isomorphism classes of (complex) vector bundles over X. It is an abelian monoid with respect to $\oplus$, and G adds formal inverses to this operation making $K(X)$ an abelian group. Moreover, $\otimes$ makes $K(X)$ a ring.

What does an element of $K(X)$ look like? Well, any vector bundle $E \to X$ sits in there as $[E]$ or simply as E. An arbitrary element looks like $[E]-[F]$, and we call anything of this form a virtual (vector) bundle. In particular, we denote the trivial bundle of rank n by $\underline{\mathbb{C}^n}$.

Remark. By a theorem of Atiyah and (independently discovered by) Jänich we have that

$$K(X) \cong [X, \mathcal{F}]$$

where $\mathcal{F}$ is the space of Fredholm operators on an infinite dimensional, complex, separable Hilbert space H (which are all isomorphic). The topology on $\mathcal{F}$ is given by the operator norm and $[X, \mathcal{F}]$ is the set of homotopy classes of maps $X \to X$. This already hints at how K-theory is weaved into the story of the Index Theorem.

Why do we want the compact Hausdorff condition on X? It comes down to the following fact: every vector bundle over a compact Hausdorff space is a direct summand of a trivial bundle. The idea here is to embed the bundle into $X \times \mathbb{C}^n$ by patching it along a finite trivializing cover of X. Once we have an embedding, we can take the orthogonal complement to give us the other direct summand.

With this fact in hand, let's look at an element of $K(X)$, say, $E - F$. We have some bundle G such that $F \oplus G = X \times \mathbb{C}^n$. Then,

$$E - F = E + G - F + G = E + G - (X \times \mathbb{C}^n) = \tilde{E} - \underline{\mathbb{C}^n}.$$

Dually, we could have gotten $E - F = \underline{\mathbb{C}^n} - \tilde{F}$. In a similar vein, we can prove that $[E] = [F]$ iff there is some n such that $E + \underline{\mathbb{C}^n} = F + \underline{\mathbb{C}^n}$.

Keeping the pullback of vector bundles in mind, we see that K is a contravariant functor, in particular, given $f : X \to Y$, we get $f^* : K(Y) \to K(X)$. In fact, K is a homotopy functor, i.e, if X is homotopy equivalent to Y, then $K(X) \cong K(Y)$.

Example 1.1. Clearly, the vector bundles over a point $(*)$ are just $\underline{\mathbb{C}^n}$. We also have homotopy invariance, so

$$K(*) \cong \mathbb{Z}.$$

Example 1.2. It turns out the every (complex) vector bundle over S^1 is trivial. One can prove this by showing that every such bundle comes from a bundle over the interval, which is contractible. Thus,

$$K(S^1) \cong \mathbb{Z}.$$

2 The clutching construction

The clutching construction is a way to glue together two vector bundles as long as you have the appropriate gluing data for the intersection of the base spaces. While it is often used in the special case of a sphere viewed as two disks glued along the edges, there is a more general construction:

Theorem 2.1. *Suppose* $X = X_0 \bigcup X_1$ *is a union of two compact spaces and let* $A = X_0 \bigcap X_1$ *such that* $X = X_0 \coprod_A X_1$*. Let* $E_i \to X_i$ *be vector bundles and* $\phi : E_0|_A \to E_1|A$ *be an isomorphism. Then, we get a bundle*

$$E_0 \cup_\phi E_1 \to X$$

We'll look at a special case where $X_1 \subset X_0$. Now, suppose we have a bundle $E \to X$ and $A \subset X$ is a closed subspace with a trivialization $\alpha : E|_A \to \underline{\mathbb{C}^n}$. Consider the equivalence relation on E given by $e \sim e'$ if $\alpha(e) = \alpha(e')$. This gives a map $E/\alpha = E/\sim \to X/A$. We have some straightforward lemmas:

Lemma 2.1. $E/\alpha \to X/A$ *is a vector bundle. Furthermore, if* $q : X \to X/A$ *is the quotient map, then* $q^*E/\alpha \cong E$*.*

The key fact to prove here is that we have a local trivialization around A/A. This follows from a fact from the theory of vector bundles, which allows us to extend an isomorphism of vector bundles on a closed set to a neighbourhood of the closed set.

Lemma 2.2. *The isomorphism class of* E/α *depends only on the homotopy class of* α*.*

Theorem 2.2. *Let* $A \subset X$ *be closed and contractible. Then, we have a bijection* $q^* : Vect(X/A) \to Vect(X)$*. In particular,* $K(q) : K(X/A) \to K(X)$ *is an isomorphism.*

Proof. Suppose $E \to X$ is a vector bundle. Since A is contractible, $E|_A$ is trivial. Any two trivializations are the same upto a composition by an automorphism of $\underline{\mathbb{C}^n}$, i.e., $GL_n(\mathbb{C})$. Since $GL_n(\mathbb{C})$ is path connected, any two trivializations are homotopic.

Thus, we get a map $Vect(X) \to Vect(X/A)$ by $E \to E/\sim$. This is an isomorphism by **Lemma 2.1**.

□

For such an isomorphism to hold for ordinary cohomology, one requires further conditions like $A \to X$ being a cofibration or A being an NDR (neighbourhood deformation retract). In this sense, K-theory acts more nicely than ordinary cohomology.

3 Homotopical and cohomological properties of K-theory

In this section, all spaces are compact Hausdorff unless specified otherwise.

We mentioned how we want to make K-Theory into a generalized cohomology theory, so in that spirit, we define the *reduced K-Theory* of a pointed space (X, x):

$$\tilde{K}^0(X) = ker(K^0(X) \to K^0(x)).$$

This gives us that $K^0(X) = \tilde{K}^0(X) \oplus \mathbb{Z}$. Similar to ordinary cohomology, we get an exact sequence

$$\tilde{K}^0(X/A) \to K(X) \to K(A)$$

using **Lemma 2.1**.

Moving on to pointed spaces, which we denote as (X, x), recall the the notions of the wedge product, smash product and cone, resp: $X \vee Y = X \times \{y\} \cup \{x\} \times Y$, $X \wedge Y = X \times Y / X \vee Y$ and $CX = X \wedge [0,1]$ (which is contractible). The mapping cone of a map of pointed spaces $f : Y \to X$ is $Cf = X \cup_f CY$.

Given an inclusion of a closed subspace $i : A \to X$, we get the Puppe sequence (an LES of pointed spaces)

$$A \to X \xrightarrow{j} Ci \xrightarrow{k} Cj \to Ck \dots$$

which induces an exact sequence

$$\dots \ \tilde{K}^0(Cj) \to \tilde{K}^0(Ci) \to \tilde{K}^0(X) \to \tilde{K}^0(A).$$

Since any cohomology theory needs to satisfy the suspension axiom, we define

$$\tilde{K}^{-n}(X) := \tilde{K}^0(\Sigma^n X).$$

If X is unpointed, then we can define

$$\tilde{K}^{-n}(X) := \tilde{K}^0(\Sigma^n X_+).$$

where X_+ is obtained by adjoining a free basepoint to X.

From this, we obtain an LES of K-Theory,

$$\tilde{K}^{-n}(X/A) \to K^{-n}(X) \to K^{-n}(A) \ \dots \ K^{-1}(A) \to \tilde{K}^0(X/A) \to K^0(X) \to K^0(A).$$

In particular, setting $A = *$ gives us that $K^{-n}(X) \cong \tilde{K}^{-n}(X) \oplus K^{-n}(*) \cong \tilde{K}^{-n}(X) \oplus K^0(S^n)$.

3.1 Relative K-Theory

From the above discussion, one might be tempted to define $K^0(X,Y)$ as $\tilde{K}^0(X,Y)$ and this indeed works. However, we also provide an alternate characterization which will be useful.

Definition 3.1. Let $Y \subset X$ be a closed subspace of a compact Hausdorff space. A K-cyckle on (X,Y) is a triple (E,F,ϕ) such that $E,F \to X$ are bundles and $\phi : E \to F$ is a bundle map with $\phi|_Y$ being an isomorphism. (E,F,ϕ) is acyclic if ϕ is an isomorphism.

An isomorphism of K-cycles $(E_0,F_0,\phi_0) \cong (E_1,F_1,\phi_!)$ is a pair of isomorphisms $(\alpha : E_0 \to E_1, \beta : f_0 \to F_1)$ such that $\phi_1 \circ \alpha = \beta \circ phi_0$.

A *concordance* between two K-cycles as above is a K-cycle on $(X \times I, Y \times I)$ such that the restriction at endpoints of the interval gives us the two K-cycles we started with.
Let $\mathbb{E}(X,Y)$ be the monoid of concordance classes and $\mathbb{D}(X,Y)$ be the monoid of concordance classes with an acyclic K-cycle. Then we define

$$K(X,Y) := \frac{\mathbb{E}(X,Y)}{\mathbb{D}(X,Y)}.$$

3.2 K-Theory with compact support

If one wants to talk about Kgroups for non-compact manifolds, we reduce it the compact case by taking the one point compactification. Thus for a locally compact Hausdorff space X, we define

$$K_c(X) := K(X^+, \infty) \cong \tilde{K}(X^+).$$

Equivalently, we could define it using K-cycles, with the slight change that for (E,F,ϕ), we want ϕ to be an isomorphism outside of a compact set. Again, we would define it as the quotient of concordance classes with acyclic ones.

Once we have that, we can define

$$K_c^{-n}(X) := K_c(X \times \mathbb{R}^n)$$

which mirrors our approach for the compact case (since $(\mathbb{R}^n)^+ \cong S^n$).

Finally, we get an LES for the compactly supported K-Theory:

$$\dots \; K_c^{-n}(X - Y) \to K_c^{-n}(X) \to K_c^{-n}(Y) \; \dots$$

With the stage set up, we can finally move...

4 Towards Bott Periodicity

The first step towards Bott Periodicity is the Bott class. Suppose we have a map $\alpha : S^{n-1} \to GL_m(\mathbb{C})$. We get a bundle homomorphism

$$D^n \times \mathbb{C}^m \to D^n \times \mathbb{C}^m \text{ givn by } (x, v) \mapsto (x, |x|\alpha(x/|x|)v).$$

This gives a map

$$\pi_{n-1}(GL_m(\mathbb{C}) \to K(D^n, S^{n-1}).$$

For $n = 2, m = 1$, the image of the conjugate map is called the *Bott class, b,* and it lives in $K(D^2, S^1)$. Alternatively, one could define it as the dual of exterior algebra of $\mathbb{C}$, $b = \lambda^*_{\mathbb{C}} \in K_c(\mathbb{C})$.

Doing the above (exterior algebra) construction for an arbitrary vector bundle $V \to X$, we get τ_V the *Thom class.* In particular, $b = \tau_{\mathbb{C}}$.

Definition 4.1. Given two K-cycles (E_0, E_1, g) and (F_0, F_1, f) on (X, A) and (Y, B) respectively, their cross product is defined as the following K-cycle on $(X \times Y, A \times B)$:

$$(E_0 \otimes F_0 \oplus E_1 \otimes F_1, E_0 \otimes F_1 \oplus E_1 \otimes F_0, f\#g := \left(\begin{smallmatrix} 1\otimes f & g^*\otimes 1 \\ g\otimes 1 & -1\otimes f^* \end{smallmatrix}\right))$$

Finally for the famed theorem:

Theorem 4.1 (Bott Periodicity)**.** *Let X be locally compact. Then the map*

$$\beta_X : K_c(X) \to K_c(X \times \mathbb{R}^2) \textit{ given by } x \mapsto x\#b$$

is an isomorphism.

We also have a generalization, which won't be too important for us, be is pretty important in its own right:

Theorem 4.2 (Thom Isomorphism)**.** *Let X be a locally compact space and $\pi : V \to X$ be a rank n bundle. Then the map*

$$th^K : K_c(X) \to K_c(V) \textit{ given by } x \mapsto \pi^* x\#\tau_V$$

Remark. Every generalized cohomology theory is represented by a spectrum, i.e, there is a sequence of spaces E_n such that $E_{n+1} \cong \Sigma E_n$ and the n^{th} cohomology group is given by $[-, E_n]$. For K-Theory, $K(X) \cong [X, BU \times Z]$, where $U = U(\infty) = colim\ U(n)$ is the infinite unitary group and BU is its classifying space. Bott periodicity implies $K^2(X)$ should be $K(X)$ and more generally that $BU \times Z \simeq \Omega^2 BU$ or equivalently, $\Omega^2 U \simeq U$ where Ω is the loop space functor. From which we get the classical result that Bott proved, i.e, $\pi_{k+2}(U) = \pi_k(U)$. There are analogous periodicity results for the real(orthogonal) and the spinor case, both of which have 8-fold periodicity. Thus the corresponding K-theories, KO-theory and KSp-theory have 8-fold periodicity as well.

4.1 A Brief Overview of the Proof of Bott Periodicity

There's a lot of work that needs to be done to prove this theorem, but out approach can basically be broken into parts: what simple conditions do we need on a map to be an inverse of β and how to find such a map.

The first is a formal algebraic "trick" due to Atiyah. We note the following facts about β :

- β_X is natural in X
- β_X is $K_c(X)$-linear with left multiplication.
- $\beta_*(1) = b$

Atiyah's trick tells us that the fairly straightforward expectations from an inverse are in fact enough to prove that it is an inverse

Theorem 4.3 (Atiyah's rotation trick)**.** *If for all compact Hausdorff X, we have* $\alpha_X : K_c(X \times \mathbb{R}^2) \to K_c(X)$ *such that*

- α_X *is natural in* X
- α_X *is* $K_c(X)$*-linear*
- $\alpha_*(b) = 1$

then we can extend α *to locally compact spaces and it is in fact the inverse of* β

The proof of extending α to locally compact spaces is in terms of the one point compactification of the space. Showing that such a map is the inverse requires a fair bit of algebraic manipulation and heavily using the ring structure of $K(X)$.

Given this, finding such an α is the hard part. The idea is to use the Toeplitz index theorem:

Theorem 4.4. *If* $f : S^1 \to \mathbb{C}^*$ *is continuous, then* T_f *is Fredholm and* $Index(T_f) = -deg(f)$.

whose baby case we've already seen for $f(z) = z^k$.

The idea then is to replace both the domain and codomain of α_X with naturally isomorphic objects which are amenable to a parametrized version of the Toeplitz Index theorem.

But where does the analysis come in? We actually want to replace the codomain of α with what is essentially a concordance class of Fredholm families, which are cycles where the bundles are Hilbert bundles and the map is locally a Fredholm operator. This is where the parametrized Toeplitz index theorem comes into play.

While we have brushed past almost all technicalities, we reiterate that a substantial amount of work has to be done to figure out the whole proof.

5 Wrapping up the story: the Index Theorem

The case we want to look at is somewhat more general than the case of Dirac operators

5.1 Elliptic Differential Operators

Definition 5.1. A *differential operator of order* k between two smooth vector bundles $E_0, E_1 \to M$ is a map $P : \Gamma(M, E_0) \to \Gamma(M, E_1)$ such that P is local and on a trivial open set, it looks like

$$f \mapsto \sum_{|I| \leq k} A^I \frac{\partial^I}{\partial x_I} f$$

where I is a multi-index. We denote the vector space of such operators by $Diff^k(E_0, E_1)$.

For $y \in M, \eta \in T_y^* M, e \in (E_0)(y)$, choose $f : M \to \mathbb{R}$ so that $f(y) = 0$ and $f^* = \eta$ and choose $s \in \Gamma(M, E_0)$ with $s(y) = e$. The *symbol* of P is defined as

$$smb_k(P)(y, \eta)(e) = \frac{i^k}{k!} P(f^k s)(y) \in E_1(y).$$

If we look at it as a function of e, then it is in fact a homogeneous polynomial of degree k in η.

We say that P is *elliptic* if for all $x \in M$, the map $smb_k(P)(\eta) : (E_0)_x \to (E_1)_x$ is invertible. In particular, every Dirac operator is elliptic.

Thus our formulation of the Index Theorem is about the index of elliptic differential operators.

Let M be a closed Riemannian manifold and $V, W \to$ be complex vector bundles and $D : \Gamma(M, V) \to \Gamma(M, W)$ be an elliptic differential operator of order k. Let DM and SM be the unit disc and sphere bundle inside the cotangent bundle $\pi : T^*M \to M$. Then, $(\pi^* V, \pi^* W, smb_k(D))$ is a K-cycle and the resulting class $\sigma(D) \in K^0(DM, SM) \cong K(T^*M)$ is called the *symbol class.*

5.2 The Index Theorem

Let M be embedded in $\mathbb{R}^n$ and U be an open tubular neighbourhood of M. Then TU is an open tubular neighbourhood of TM in $T\mathbb{R}^n$. The *topological index, tind,* is the map $K_c(TM) \to \mathbb{Z}$ defined as

$$tind : K_c(TM) \xrightarrow{th^K} K_c(TU) \to K_c(\mathbb{R}^{2n} = T\mathbb{R}^n) \overset{Bott}{\cong} \mathbb{Z}$$

Then, the Index Theorem is -

Theorem 5.1 (Atiyah-Singer Index Theorem)**.**

$$ind(D) = tind(\sigma(D))$$

One can look at the index as a map from the K-theory, which sends the symbol to the index, so the Index Theorem can be restated as the fact that the analytic index and the topological index as maps from $K_c(TM) \to \mathbb{Z}$ are the same map. This is a really pleasant and succinct way of looking at the the Index theorem! If $M = \mathbb{R}^2$, the theorem just boils down to Bott Periodicity.

Remark. There's another approach where we define K-homology and the Index Theorem becomes a statement regarding the pairing of K-homology and K-Theory which is the K-theoretic version of Poincaré duality! In some sense, this is a very natural way through which we can look at the Index Theorem. A more general way to tackle this is via the K-theory and K-homology of C^*-algebras which leads to some powerful generalizations.

How does this relate to the cohomological version of the Index Theorem? By using the chern character! We get the following diagram:

$$\begin{array}{ccccccc}
K_c(TM) & \xrightarrow{th_K} & K_c(TU) & \longrightarrow & K_c(T\mathbb{R}^n) & & \\
{\scriptstyle ch\circ\mu(\nu_{\mathbb{C}})}\big\downarrow & & \big\downarrow{\scriptstyle ch} & & \big\downarrow{\scriptstyle ch} & \searrow^{\beta^{-n}} & \\
H_c^*(TM) & \xrightarrow{th_H} & H_c^*(TU) & \longrightarrow & H_c^*(T\mathbb{R}^n) & \xrightarrow{\int} & \mathbb{R}
\end{array}$$

Where th_H is the cohomological version of the Thom map, H_c^* is cohomology with compact supports and μ is a factor added to make sure that the diagram commutes.
From this, we get

$$tind(x) = \int_{TM} ch(x)\mu(\nu_M \otimes \mathbb{C}).$$

μ is nothing but the Todd class, and further, if M is oriented and we let $x = \sigma(D)$, we can reduce it to

$$ind(D) = tind(D) = (-1)^{n(n-1)/2} \int_M th_H^{-1}(ch(\sigma(D)))Td(TM \otimes \mathbb{C}).$$

Thus we see that the Todd class rears its head purely as an error factor while converting a natural statement about K-Theory to mildly clumsy statement about ordinary cohomology!

6 References

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