Connections on Vector Bundles and Characteristic Classes

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0 Introduction

Characteristic classes lie at the intersection of geometry and topology. Essentially, they are natural cohomology classes of vector bundles and thus give algebraic invariants of geometry. They can also be used to figure out information about the orientaion and spin structure on manifolds (though we will not go into that).

There are multiple approaches to characteristic classes:

- One can define them axiomatically à la Grothendieck
- The homotopical approach is to define them using classifying spaces.
- The geometric approach (the one we will use) uses connections to define said classes.

One of the the neat parts of the geometric approach is that the characteristic class thus defined is independent of the connection chosen on the vector bundle, as one would expect if we want to do something truly topological.

We'll develop the theory for real vector bundles and give a brief note regarding the complex case and Chern classes. There is a more general theory for principal G-bundles, which we'll will state.

A good reference for the complex case is the appendix of Milnor and Stasheff's *Characteristic classes*. This article mainly follows Loring Tu's *Differential Geometry.*

1 Vector Bundles

Everything in sight will be smooth. We will denote $C^\infty(M)$ by $\mathcal{F}$.

Definition 1.1. A *vector bundle of rank n* over $\mathbb{R}$ is a surjective smooth map $\pi : E \to M$ such that

i) For every $p \in M,\ E_p := \pi^{-1}(p)$ is a vector space of rank n

ii) For every $p \in M$, there exists an open $U \subset M$ such that there is a fiber-preserving diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbb{R}^n$ that restricts to a linear isomorphism on each fiber.

We call M the base space and E the total space. There are a bunch of examples familiar to us:

Example 1.1. The trivial bundle $M \times \mathbb{R}^n \to M$.

Example 1.2. The tangent bundle $TM \to M$ and the cotangent bundle $T^*M \to M$

Essentially, a vector bundle is a family of vector spaces parametrized by a manifold M. We naturally want to talk about maps of vector bundles.

Definition 1.2. A *bundle map* between two vector bundles $\pi_E : E \to M$ and $pi_F : F \to N$ is a pair of smooth maps $(\phi : E \to F, \bar{\phi} : M \to N)$ such that $\bar{\phi} \circ \pi_E = \pi_F \circ \phi$ and ϕ restricts to a linear map on the fibers.

We will often encounter the case where $M = N$ and $\bar{\phi} = id_M$. A *section* $s : M \to E$ of a vector bundle $\pi : E \to M$ is a smooth map such that $\pi \circ s = id_M$. One should think of a section as picking out a choice of elements in the fibers in a smooth manner. We can also talk about sections over an open subset U of M.

Example 1.3. Sections of $TM \to M$ are vector fields, which we denote by $\mathfrak{X}(M) = \Gamma(TM)$. We also have the vector bundle $\bigwedge^k T^*M \to M$, whose sections are *k-forms.*

The set of sections $\Gamma(E, U)$ is naturally a vector space over $\mathbb{R}$, as it inherits the operations from the fibers. Moreover, it is a module over $C^\infty(U)$, where a function f acts by point-wise multiplication.

We can also see that a map of bundles $\phi : E \to F$ over M induces a map on the sections $\phi_\# : \Gamma(E, M) = \Gamma(E) \to \Gamma(F)$ by composition. This is in fact an $\mathcal{F}$-linear map. One can show that this is a bijection $\{bundle\ maps\ E \to F\} \leftrightarrow \{\mathcal{F} - linear\ maps\ \Gamma(E) \to \Gamma(F)\}$. To see that # is surjective, we just define a bundle map fibre-wise using the map on the sections. Injectivity follows from the fact that for any $p \in M, e \in E_p$, there is a section s such that $s(p) = e$, which can be constructed using bump functions.

Definition 1.3. A *frame* for a vector bundle of rank n for an open set U in the base space is a collection of sections $e_1, \dots, e_n$ over U such that $e_1(p), \dots, e_n(p)$ is a basis of the fiber of $p \in U$.

A frame over the base space exists if and only if the bundle is trivial. In particular, there always exists a frame over a trivializing open set.

2 Connections on Vector Bundles

Definition 2.1. A connection on a vector bundle $E \to M$ is a map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$$

such that for all $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$, we have
i) ∇ is $\mathcal{F}$-linear in the first variable and $\mathbb{R}$-linear in the second.
ii)

$$\nabla_X(fs) = (Xf)s + f\nabla_X s$$

for all $f \in \mathcal{F}$. That is,

$$\nabla_X(fs) = (df)(X)s + f\nabla_X s.$$

Slightly abusing notation, we can rewrite it as

$$\nabla(fs) = df \cdot s + f\nabla_s.$$

In particular, (affine) connections on manifolds are just connections on $TM \to M$.
We can define a connection on a trivial bundle by specifying that it is 0 at a particular frame and then extending linearly by the Leibniz Rule (ii above).

Analogous to the case of affine connections, we can show that every vector bundle has a connection. To do this, we define it on trivializing open sets (as above), and then extend to the bundle by patching them up using a partition of unity subordinate to the cover of trivializing open sets. Note that if ∇^i are connections, then so is $\sum \lambda_i \nabla^i$ where $\sum \lambda_i = 1$.

Curvature for connections on vector bundles is pretty similar to the affine case:

$$R(X,Y) = \nabla_X \nabla_Y + \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Definition 2.2. A *Reimannian metric* on a bundle $\pi : E \to M$ is an inner product on each fiber E_p, which is smooth in the sense that $\langle s, t \rangle$ is a smooth map $M \to \mathbb{R}$ for any two sections $s, t \in \Gamma(E)$.

In particular, a Riemannian metric on M is just a Reimannian metric on $TM \to M$. A bundle with a metric is called a Riemannian bundle. Every bundle can be imbued with a metric, and the proof of this is similar to the proof about connections above.

A connection ∇ on a Riemannian bundle is said to be compatible with the metric if for all $X \in \mathfrak{X}(M),\ s, t \in \Gamma(E)$

$$X\langle s, t\rangle = \langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle.$$

The connection on a trivial bundle defined above is compatible with the natural metric on the bundle.

Similar to the previous case, every Riemannian bundle has a compatible metric. The key fact needed to prove this is that an $\mathcal{F}$-linear sum of compatible connections is a compatible connection when the sum of the coefficients is 1.

Consider some connection ∇ on $\pi : E \to M$. Let U be a trivializing open set with a frame $e_1, \ldots, e_n$. Then, we can write ∇ locally as

$$\nabla_X e_j = \sum \omega_j^i(X) e_i$$

We call $\omega = [\omega_j^i]$ the *connection matrix* of ∇ with respect to $(U, (e_1, \ldots, e_n))$. Note that each entry in the matrix is a 1-form

Similarly, we can look at the curvature $R(X, Y)$ locally. We get a matrix $\Omega = [\Omega_j^i]$ of 2-forms which we call the *curvature matrix.*

Theorem 2.1.

$$\Omega_j^i = d\omega_j^i + \sum_k \omega_i^k \wedge \omega_j^k$$

More succinctly,

$$\Omega = d\omega + \omega \wedge \omega.$$

The proof is fairly straightforward and just involves expanding the definitions and rewriting.

We obtain the following corollary by differentiating both sides.

Corollary 2.1 (The second Bianchi Identity)**.**

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$$

.

We can rewrite the definition of the connection matrix as (using matrix notation):

$$\nabla e = e\omega.$$

Suppose we have a different frame $\bar{e}$ for U. We would like to know what happens to the connection matrix and the curvature matrix under this change of frame.

Let $a = [a_l^k]$ be the change of frame matrix from e to $\bar{e}$. Then, the corresponding curvature and connection matrices are:

$$\bar{\omega} = a^{-1}\omega a + a^{-1}da.$$

$$\bar{\Omega} = a^{-1}\Omega a.$$

Where the differential operator d acts entry-wise on a.

We now come to the question of building a connection using connection matrices. Suppose we have a trivializing cover (U_α, e^α) of M with some frames and corresponding connection matrices ω_α. Then, they form a connection on the whole bundle iff on each pairwise intersection $U_\alpha \cap U_\beta$ the change of frame identity is satisfied for ω_α and ω_β.

This helps us with constructing connections on bundles where we only have local information. Suppose we have a bundle $\pi : E \to M$ and a map $f : M \to N$. Then the *pullback bundle* over N is defined by

$$f^*(E) = \{(n, e) \in N \times E : f(n) = \pi(e)\}.$$

and the bundle map is just projection onto N. Suppose we have a connection ∇ on π. Now that we can work with local information, we can pull back ω_α as its entries are just 1-forms and this gives us connection matrices which satisfy the compatibility condition above. Thus we can pull back a connection to the pullback bundle.

Let's move on to the setting of Riemannian bundles again. What restrictions does a compatible connection impose on the connection matrices? Does every family of connection matrices satisfying the above property come from a compatible connection? The answer is quite nice.

Theorem 2.2. *Let ∇ be a connection on a Riemannian bundle $\pi : E \to M$. Then,*

(i) If the connection is compatible then the connection matrix ω for any open set U with an orthonormal frame is skew-symmetric.

(ii) If for all $p \in M$, we have a trivializing open set U with an orthonormal frame such that ω is skew-symmetric, then ∇ is compatible with the metric.

Proof. (i) For any $X \in \mathfrak{X}(M)$ and i, j,

$$0 = X\langle e_i, e_j\rangle = \langle \nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle = \langle \omega_i^k(X) e_k, e_j\rangle + \langle e_i, \omega_j^k(X) e_k\rangle = \omega_i^j + \omega_j^i.$$

(ii) It is enough to show that the connection is compatible locally. Let ω be skew-symmetric and $s = a^i e_i, t = b^j e_j$. Skipping some of the steps, we have

$$X\langle s, t\rangle = X(\sum a^i b^i) = \sum (Xa^i) b^i + \sum (a^i (Xb^i).$$

$$\begin{aligned} \langle \nabla_X s, t\rangle &= \sum (Xa^i) b^i + \sum a^i \omega_i^j(X) b^i. \\ \langle s, \nabla_X t\rangle &= \sum (Xb^i) a^i + \sum b^i \omega_i^j(X) a^j. \\ &= \sum (Xb^i) a^i + \sum a^i b^j \omega_j^i(X) \end{aligned}$$

Adding them up,

$$\langle \nabla_X s, t\rangle + \langle s, \nabla_X t\rangle = \sum (Xb^i) a^i + (Xa^i) b^i = X\langle s, t\rangle.$$

□

Theorem 2.1 tells us that if the connection matrix is skew-symmetric, then so is the curvature matrix.

3 Characteristic classes

We've seen that $\bar{\Omega} = a^{-1}\Omega a$. Thus if we define some function (using Ω) that is invariant under conjugation by elements of $GL(\mathbb{R}, n)$, then that function will be independent of the the frame chosen to represent the connection ∇. This is our preliminary strategy to define characteristic classes.

Definition 3.1. Let $X = [x^i_j]$ be a matrix of indeterminates. Let $P(X)$ be a polynomial over $\mathfrak{gl}(\mathbb{R}, n) = \mathbb{R}^{n\times n}$ with variables from X. Then $P(X)$ is said to be *invariant* if for all $A \in GL(\mathbb{R}, n)$

$$P(X) = P(A^{-1}XA).$$

It is in fact enough if the above holds for all real matrices X. More generally, this implies that the identity holds for all matrices with entries in some $\mathbb{R}$-algebra $\mathcal{A}$.

Example 3.1. det(X) and tr(X) are invariant polynomials.

Example 3.2. Let $det(\lambda I + X) = \sum \lambda^i f_{n-i}(X)$. Then each f_i is an invariant polynomial.

Example 3.3. The polynomials $\sigma_k(X) := tr(X^k)$ are invariant polynomials.

Let P be an invariant polynomial of degree k. Consider any $p \in M$. Let

$$\mathcal{A} = \bigoplus_i \bigwedge^{2i}(T^*_p M).$$

Then,

$$P(\bar{\Omega}_p) = P(a^{-1}(p)\Omega_p a(p)) = P(\Omega_p).$$

Thus, $P(\Omega) \in \Omega^{2k}(U)$ is independent of the frame. Hence we can unambiguously define $P(\Omega)$ as a global 2-form on M as we know it is locally well-defined and there's no issue on intersections.

The following theorem is the heart of this article:

Theorem 3.1. *Let $\pi : E \to M$ be a vector bundle of rank n and ∇ be a connection on this bundle. Suppose we have an invariant homogeneous polynomial P of degree k on $\mathfrak{gl}(\mathbb{R}, n)$. Then,*

(i) The 2k-form $P(\Omega)$ is closed.

(ii) $[P(\Omega)] \in H^{2k}(M)$ is independent of ∇.

We have the following algebra homomorphism

$$c_E : Inv(\mathfrak{gl}(\mathbb{R}, n)) \to H^*(M)$$

$$P \mapsto [P(\Omega)]$$

This is known as the *Chern-Weil homomorphism.* Such a $[P(\Omega)]$ is known as a *characteristic class.*

Let's start with the proof of *(i)*. We'll need the following facts.

Lemma 3.1.
$$\sum_i (-1)^i f_i \sigma_{k-i} = 0$$

Which can be used to prove

Theorem 3.2.
$$Inv(\mathfrak{gl}(\mathbb{R}, n)) = \mathbb{R}[f_1, \ldots, f_n] = \mathbb{R}[\sigma_1, \ldots, \sigma_n].$$

Thus, it is enough to show that $[\sigma_k(\Omega)]$ is a closed form.

Proof of ***Theorem 3.1*** *(i).* Let e be some frame for an open set U and let ω and Ω be the respective connection and curvature matrices. Then,

$$d(\sigma_k(\Omega)) = tr(d(\Omega^k)).$$

Applying the second Bianchi identity,

$$d(\sigma_k(\Omega)) = tr(\Omega^k \wedge -\omega \wedge \Omega^k) = tr(\Omega^k \wedge \omega) - tr(\omega \wedge \Omega^k) = 0.$$

□

Let J be an open interval in $\mathbb{R}$. We can consider a smooth family of k-forms ω_t. We define the derivative of this wrt t to be

$$\left(\frac{d\omega_t}{dt}\right)_p = \dot{\omega}_t = \frac{d}{dt}\omega_{t,p}.$$

We similarly define integrals for a smooth family of forms. We can extend this definition to a matrix of forms, just by doing it entry-wise.

Lemma 3.2. *Let ω and τ (suppressing the subscript) be matrices of smooth families of differential forms. Then*

(i) $\frac{d}{dt}(tr\ \omega) = tr(\frac{d\omega}{dt})$

(ii) $\frac{d}{dt}(\omega \wedge \tau) = \dot{\omega} \wedge \tau + \omega \wedge \dot{\tau}$

(iii) $\frac{d}{dt}(d\omega) = d(\frac{d}{dt}(\omega))$

(iv) $\int_a^b d\omega\ dt = d(\int_a^b \omega\ dt)$

We will need the above lemma to fill in the gaps of the proof below.

Sketch of proof of ***Theorem 3.1*** *(ii).* Let ∇^0 and ∇^1 be two connections connections on $\pi : E \to B$. We then have a family of connections

$$\nabla^t = t\nabla^0 + (1-t)\nabla^1.$$

The key (somewhat non-trivial) fact needed for this proof is

$$\frac{d}{dt}(tr\ \Omega_t^k) = d(k\ tr(\Omega_t^{k-1}\dot{\omega}_t))).$$

Integrating both sides and simplifying, we get

$$tr(\Omega_1^k) - tr(\Omega_0^k) = d\int_0^1 k\ tr(\Omega_t^{k-1}\dot{\omega}_t)\ dt.$$

Thus $[tr(\Omega_1^k)] = [tr(\Omega_0^k)]$. We conclude that the characteristic class is independent of the connection. □

It is a consequence of **Theorem 2.2** that the characteristic classes wrt to any odd degree invariant polynomial is 0.

We claimed earlier that characteristic classes are *natural*. What we mean by that is that it is a natural transformation c from $Vect_n$ (which associates to a manifold the isomorphism classes of rank n vector bundles over it) to H^*. What this boils do is that c commutes with pullbacks. That is, given a map of smooth manifolds $f : M \to N$,

$$c_N(f^*E) = f^*c_M(E).$$

3.1 Pontrjagin Classes

Definition 3.2. The classes $p_k(E) = [f_{2k}(\frac{i}{2\pi}\Omega)] \in H^{4k}(M)$ are called *Pontrjagin classes.*

They give us information about all the characteristic classes since they generate $Inv(\mathfrak{gl}(\mathbb{R}, n))$. The factor of $\frac{i}{2\pi}$ ensures that these classes behave nicely when integrated.

The *total Pontrjagin class* of E is defined to be

$$p(E) = det\left(I + \frac{i}{2\pi}\Omega\right) = 1 + p_1 + \cdots + p_{\lfloor n/2 \rfloor}.$$

Suppose now that M is compact and orientable and E has dimension $4m$ for some $m \in \mathbb{N}$. Let a_i be numbers such that $\prod p_i^{a_i}$ is a cohomology class of degree $4m$. Then the number

$$\int_M \prod p_i^{a_i}$$

is called the *Pontrjagin number* of E. If $E = TM$, then this is a topological invariant of M.

3.2 Orientation on a vector bundle and Euler Classes

An *orientation* on a vector bundle $\pi : E \to M$ should be though of as a smooth assignment of orientation to each fiber. More precisely, it is an equivalence class of sections of the line bundle $\bigwedge^n E$ (the fiber of p here is the collection of orientations on E_p as a vector space). Two sections s and t are equivalent if there exists a positive function f such that $s = ft$. For this section, we will only consider frames which are compatible with the orientation on the vector bundle. We call these *positively oriented* frames.

Let $\pi : E \to M$ be an oriented Riemannian bundle with a compatible connection ∇. For positively oriented frames, the change of frames matrix is special orthogonal.

In this setting, we'd like to alter our definition of a characteristic class. We now want P to be a polynomial which is invariant under conjugation by elements of $\mathrm{SO}(n)$. We call such a polynomial $\mathrm{Ad}(\mathrm{SO}(n))$-invariant.

What are the generators of $Inv(\mathfrak{so}(n))$? For n odd, it is generated either by $\{\sigma_i\}$ or $\{f_i\}$. For n even, we need an additional generator: the pfaffian, which we denote by $\mathrm{Pf}(X)$ (which we know is the square root of the determinant).

The class $e(E) = [\mathrm{Pf}(\frac{1}{2\pi}\Omega)] \in H^{2m}(M)$ is called the Euler class. As before, this is a closed global form whose cohomology class is independent of the connection.
We'd like to call

$$\int_M \mathrm{Pf}\left(\frac{1}{2\pi}\Omega\right)$$

the *Euler number*. But does it agree with the traditional notion of the Euler characteristic $\chi(M)$? Indeed, they are equal when the bundle in question is the tangent bundle, and that is the statement of the generalized Gauss-Bonnet Theorem.

3.3 Chern Classes

The setting here is of complex vector bundles and Hermitian metrics. The definitions and the theory are analogous. We now want our polynomials to be invariant under conjugation by elements of $GL(\mathbb{C}, n)$. Then, we get

$$det\left(I + \frac{i}{2\pi}\Omega\right) = 1 + c_1(E) + \cdots + c_n(E).$$

We call $c_i(E)$ the i^{th} *Chern class* of E.

4 Principal G-Bundles

We've seen multiple notions of characteristic classes which vary depending on the group action we want our polynomials to be under. This begs the generalization to case of general lie groups.

Definition 4.1. Let G be a group. Then a (smooth) principal G-bundle is a map $\pi : P \to M$ that is a fiber bundle with a smooth, free right-action of G on P such that the local trivializations are G-equivariant. That is, given a smooth free action on P, we need

(i) π is surjective and $\pi^{-1}(p) \cong G$.

(ii) For each $p \in M$, there is an open U containing p with homeomorphisms

$$\phi_U : \pi^{-1}(U) \to U \times G$$

such that $\phi_U(x) \cdot g = \phi_U(xg)$

Let G now be a Lie group with lie algebra $\mathfrak{g}$. We want to generalize the notion of a connection. This is a bit complicated, but essentially it is a smooth $\mathfrak{g}$-valued 1-form satisfying some equivariance conditions. In this case, they are called *Ehresmann connections.*

Given such a connection ω. The curvature is the $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

This brings us to the final theorem, which is a generalization of **Theorem 3.1**.

Theorem 4.1. *Let Ω be the curvature with respect to a connection ω on a principal G-bundle $\pi : P \to M$ and f be an $Ad(G)$-invariant polynomial of degree k on $\mathfrak{g}$. Then,*

(i) There exists a $2k$-form Λ on M such that $f(\Omega) = \pi^*(\Lambda)$.

(ii) Λ is a closed form.

(iii) The cohomology class $[\Lambda]$ is independent of the connection.

As before, we get a homomorphism

$$w : Inv(\mathfrak{g}) \to H^*(M)$$

$$f \mapsto [\Lambda]$$

which we call the *Chern-Weil* homomorphism.